Option pricing by Esscher transforms in the cases of normal inverse Gaussian and variance gamma processes

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Abstract: The class of Esscher transforms is an important tool for option pricing Gerber and Shiu (1994) showed that the Esscher transform is an efficient technique for valuing derivative securities if the log returns of the underlying securities are governed by certain stochastic processes with stationary and independent increments. Levy processes are the processes of such type. Special cases of the Levy processes such as the normal inverse Gaussian process and the variance gamma process are considered at this paper. Values of these processes parameters for the existence of Esscher transform are deduced. A new algorithm of a normal inverse Gaussian process and variance gamma process simulation is also presented in this paper. These algorithm is universal and simpler one compared with analogous algorithms.

Key words: Esscher transforms, option pricing, generalized hyperbolic process, normal inverse Gaussian process, variance gamma process

1 Option pricing by Esscher transforms

Let $L = (L_t)_{t \geq 0}$ be a Levy process on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_{t \geq 0}), P)$. Consider a stock price model

$$S_t = S_0 \exp(rt) \exp(L_t), \tag{1}$$

with a stock price process $S = (S_t)_{t \geq 0}$, $S_0 > 0$, a constant deterministic interest rate $r > 0$ and Levy process $L = (L_t)_{t \geq 0}$.

We cite the following definition and auxiliary results from S. Raible ([2], Chap. 1).

Definition 1 [2]. Let $L=(L_t)_{t\geq 0}$ be a Levy process on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_{t\geq 0}), P)$. We call Esscher transform any change of P to a locally equivalent measure $\widetilde{P}$ with a density process $Z_t = \left.\frac{d\widetilde{P}}{dP}\right| \mathcal{F}_{t\geq 0}$ of the form

$$Z_t = \frac{\exp(u L_t)}{[M(u)]^t}, \tag{2}$$

where $M(u)=E[e^{uL_1}]$ is the moment generating function of $L_1, u \in \mathbb{R}$.

Lemma 1 [2]. Let $Z=(Z_t)_{t\geq 0}$ be a density process, i.e. a non-negative P-martingale with $E[Z_t]=1$ for all t. Let $\widetilde{P}$ be the measure defined by $\left.\frac{d\widetilde{P}}{dP}\right| \mathcal{F}_t = Z_t$, $t \geq 0$. Then an adapted process $X=(X_t)_{t\geq 0}$ is a $\widetilde{P}$-martingale iff $(X_t Z_t)_{t\geq 0}$ is a P-martingale.

If further we assume that $Z_t > 0$ for all $t \geq 0$, we have the following. For any $t < T$ and any $\widetilde{P}$-integrable and $\mathcal{F}_T$-measurable random variable Y,

$$E_{\widetilde{P}}[Y \mid \mathcal{F}_t] = E_P\left[Y \frac{Z_T}{Z_t} \,\middle|\, \mathcal{F}_t \right] \tag{3}$$

Lemma 2 [2]. Equation (2) defines a density process for all $u \in \mathbb{R}$ such that $E[\exp(uX_1)] < \infty$. $L=(L_t)_{t\geq 0}$ is again a Levy process under the new measure $\widetilde{P}$.

Lemma 3 [2]. Let the stock price process be given by (1). The random variable L_1 is non-degenerate and possesses a moment generating function $M(u): u \mapsto E[e^{uL_1}]$ on some open interval (a,b), $a,b \in \mathbb{R}$ with $b-a>1$ and there exists a real number $u \in (a, b-1)$ such that

$$M(u) = M(u+1). \tag{4}$$

Then the basic probability measure P is a locally equivalent to a measure $\widetilde{P}$ such that the discounted stock price $\exp(-rt)S_t = S_0 \exp(L_t)$ is a $\widetilde{P}$-martingale. A density process leading to such a martingale measure $\widetilde{P}$ is given by the Esscher transform density (2) with a suitable real u. The value u is uniquely determined as the solution of (4).

Raible [2] introduces the certain example which shows that equation (4) doesn't hold for any $u \in (a, b-1)$ and any Levy process parameters. We find out the values $u \in (a, b-1)$ and the Levy process parameters values in order to equation (4) holds.

Let the Levy process of the model (1) be the normal inverse Gaussian process $L^{NIG} = \left(L_t^{NIG}\right), t \geq 0$ with a shape parameter $\alpha > 0$, a skewness parameter $-\alpha < \beta < \alpha$, a scale parameter $\delta > 0$ and a location parameter $\mu \in \mathbb{R}$.

Definition 2. A stochastic process $L^{NIG} = \left(L_t^{NIG}\right), t \geq 0$ with the parameters $\alpha, \beta, \delta, \mu$ on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_{t\geq 0}), P)$ having values in $\mathbb{R}$, is called the normal inverse Gaussian process if

1) $L_0^{NIG} \overset{a.s.}{=} 0$;
2) L^{NIG} has independent and stationary increments;
3) L^{NIG} increment $L_{t+s}^{NIG} - L_s^{NIG}$ follows a normal inverse Gaussian law with the parameters $\alpha > 0, -\alpha < \beta < \alpha, \delta t > 0, \mu t \in \mathbb{R}$, i.e.

$$L_{t+s}^{NIG} - L_s^{NIG} \overset{D}{=} L_t^{NIG} - L_0^{NIG} \sim NIG(\alpha, \beta, \delta t, \mu t),\ s \geq 0, t \geq 0,$$

The normal inverse Gaussian process was introduced by Barndorff-Nielsen (1998) as a stock price model [3].

The normal inverse Gaussian distribution characteristic function is given by

$$\varphi(u) = \exp(\mu u) \frac{\exp\left(\delta\sqrt{\alpha^2 - \beta^2}\right)}{\exp\left(\delta\sqrt{\alpha^2 - (\beta + iu)^2}\right)}$$

Hence the moment generating function $M(u) = \varphi(-iu)$ of a normal inverse Gaussian law is of the form

$$M(u) = \exp(\mu u) \frac{\exp\left(\delta\sqrt{\alpha^2 - \beta^2}\right)}{\exp\left(\delta\sqrt{\alpha^2 - (\beta + u)^2}\right)}. \tag{5}$$

Theorem 1. There is the only point $u = -\frac{1}{2} - \beta$, $u \in (-\alpha - \beta, \alpha - \beta)$ such that (4) holds when the normal inverse Gaussian distribution parameters α, β, δ, $\mu = 0$ satisfy the following conditions $\alpha > \frac{1}{2}$, $-\alpha < \beta < \alpha$, $\delta > 0$.

Proof. The moment generating function (5) range of definition is $[-\alpha - \beta, \alpha - \beta]$, so the maximal interval on which the moment generating function exists is $(-\alpha - \beta, \alpha - \beta)$. According to assumption $b - a > 1$, where $b = \alpha - \beta, a = -\alpha - \beta$, we deduce that $\alpha > \frac{1}{2}$.

Let’s find such $u \in (a, b-1)$ that (4) holds. We have the following equation from (4), (5)

$$\frac{\exp\left(\delta\sqrt{\alpha^2-\beta^2}\right)}{\exp\left(\delta\sqrt{\alpha^2-(\beta+u)^2}\right)}=\frac{\exp\left(\delta\sqrt{\alpha^2-\beta^2}\right)}{\exp\left(\delta\sqrt{\alpha^2-(\beta+u+1)^2}\right)}. \qquad (6)$$

Having solved (6) we get $u=-\frac{1}{2}-\beta$.

Confirm theorem 1 by the following example.

Example 1. Consider normal inverse Gaussian distributions with parameters $\alpha=1$, $\beta=-0{,}1$, $\delta=0{,}005$, $\mu=0{,}001$ in the first case and $\alpha=1$, $\beta=-0{,}1$, $\delta=0{,}005$, $\mu=0$ in the second case. The moment generating function (5) range of definition in both cases is $\left[-\alpha-\beta,\alpha-\beta\right]=\left[-0{,}9{,}1{,}1\right]$, so the maximal open interval on which the moment generating function exists is $\left(-0{,}9{,}1{,}1\right)$. However in the first case there are no two points $u, u+1$ in the range of definition such that the values of the moment generating function at these points are the same. In the second case there are two points $u=-\frac{1}{2}-\beta=-0{,}4$, $u+1=0{,}6$ in the range of definition such that the values of the moment generating function at these points are the same $M(u)=M(u+1)=1{,}000645$.

Let the Levy process of the model (1) be the variance gamma process $L^{VG}=\left(L_t^{VG}\right), t\ge 0$ with a shape parameter $\alpha>0$, a skewness parameter $-\alpha<\beta<\alpha$, a scale parameter $\delta>0$ and a location parameter $\mu\in\mathbb{R}$. The variance gamma process was introduced by Madan D.B. et al. [4] (1998) as a stock price model.

Definition 3. A stochastic process $L^{VG}=\left(L_t^{VG}\right), t\ge 0$ with the parameters $\sigma>0, \nu>0, \theta\in\mathbb{R}$ on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_{t\geq 0}), P)$ having values in $\mathbb{R}$, is called the variance gamma process if

1) $L_0^{VG}\overset{a.s.}{=}0$;

2) L^{VG} has independent and stationary increments;

3) L^{VG} increment $L_{t+s}^{VG}-L_s^{VG}$ follows a variance gamma law with the parameters $\sigma>0, \nu>0, \theta\in\mathbb{R}$, i.e.

$$L_{t+s}^{VG}-L_s^{VG}\overset{D}{=}L_t^{VG}-L_0^{VG}\sim VG(\sigma\sqrt{t},\nu/t,t\theta),\ s\ge 0, t\ge 0 .$$

The variance gamma law characteristic function is given by [4]

$$\varphi(u) = \left(1 - i\theta\nu u + (\sigma^2\nu/2)u^2\right)^{-1/\nu}. \quad (7)$$

The variance gamma law moment generating function $M(u) = \varphi(-iu)$ is of the form

$$M(u) = \left(1 - \theta\nu u - (\sigma^2\nu/2)u^2\right)^{-1/\nu}. \quad (8)$$

Theorem 2. There is such point

$$u = -\frac{\theta}{\sigma^2} - \frac{1}{2},\ u \in \left(-\frac{\theta}{\sigma^2} - \sqrt{\frac{\theta^2}{\sigma^4} + \frac{2}{\sigma^2\nu}}, -\frac{\theta}{\sigma^2} + \sqrt{\frac{\theta^2}{\sigma^4} + \frac{2}{\sigma^2\nu}}\right),$$

that (4) holds, when the variance gamma law parameters σ, ν, θ satisfy the following conditions $\sigma > 0$, $\theta \in \mathbb{R}$, $\nu = 2k, k = 1,2,...$ or $\nu = 1/2k, k = 1,2,...$, and $\sqrt{\frac{\theta^2}{\sigma^4} + \frac{2}{\sigma^2\nu}} > \frac{1}{2}$.

If $\nu = 2k+1, k = 1,2,...$ or $\nu = 1/(2k+1), k = 1,2,...$, then

$$u \in \left(-\infty, -\frac{\theta}{\sigma^2} - \sqrt{\frac{\theta^2}{\sigma^4} + \frac{2}{\sigma^2\nu}}\right) \cup$$

$$\cup \left(-\frac{\theta}{\sigma^2} - \sqrt{\frac{\theta^2}{\sigma^4} + \frac{2}{\sigma^2\nu}}, -\frac{\theta}{\sigma^2} + \sqrt{\frac{\theta^2}{\sigma^4} + \frac{2}{\sigma^2\nu}}\right) \cup$$

$$\cup \left(-\frac{\theta}{\sigma^2} + \sqrt{\frac{\theta^2}{\sigma^4} + \frac{2}{\sigma^2\nu}}, +\infty\right),$$ value $u = -\frac{\theta}{\sigma^2} - \frac{1}{2}$.

Proof. If $\nu = 2k, k = 1,2,...$ $(\nu = 1/2k, k = 1,2,...)$, then the moment generating function (8) range of definition is $\left(-\frac{\theta}{\sigma^2} - \sqrt{\frac{\theta^2}{\sigma^4} + \frac{2}{\sigma^2\nu}}, -\frac{\theta}{\sigma^2} + \sqrt{\frac{\theta^2}{\sigma^4} + \frac{2}{\sigma^2\nu}}\right)$. Because of condition $b - a > 1$, where $b = -\frac{\theta}{\sigma^2} + \sqrt{\frac{\theta^2}{\sigma^4} + \frac{2}{\sigma^2\nu}}, a = -\frac{\theta}{\sigma^2} - \sqrt{\frac{\theta^2}{\sigma^4} + \frac{2}{\sigma^2\nu}}$ we have $\sqrt{\frac{\theta^2}{\sigma^4} + \frac{2}{\sigma^2\nu}} > \frac{1}{2}$.

Because of (4), (8) we have

$$\left(1 - \theta\nu u - (\sigma^2\nu/2)u^2\right)^{-1/\nu} =$$

$$= \left(1 - \theta\nu(u+1) - (\sigma^2\nu/2)(u+1)^2\right)^{-1/\nu} \quad (9)$$

Having solved (10) we get $u=-\frac{\theta}{\sigma^2}-\frac{1}{2}$.

Если $\nu=2k+1, k=1,2,...$ $(\nu=1/(2k+1), k=1,2,...)$, then the moment generating function (8) range of definition is

$$\left(-\infty,-\frac{\theta}{\sigma^2}-\sqrt{\frac{\theta^2}{\sigma^4}+\frac{2}{\sigma^2\nu}}\right)\cup$$

$$\cup\left(-\frac{\theta}{\sigma^2}-\sqrt{\frac{\theta^2}{\sigma^4}+\frac{2}{\sigma^2\nu}},-\frac{\theta}{\sigma^2}+\sqrt{\frac{\theta^2}{\sigma^4}+\frac{2}{\sigma^2\nu}}\right)\cup$$

$$\cup\left(-\frac{\theta}{\sigma^2}+\sqrt{\frac{\theta^2}{\sigma^4}+\frac{2}{\sigma^2\nu}},+\infty\right)$$. The value $u=-\frac{\theta}{\sigma^2}-\frac{1}{2}$ is deduced from (10).

Confirm theorem 2 by the following example.

Example 2. Consider variance gamma distributions with parameters $\sigma=0{,}1$, $\nu=2$, $\theta=-0{,}01$ in the first case and $\sigma=10$, $\nu=-0{,}01$, $\theta=0$ in the second case. The moment generating function (5) range of definition in the first case is $\left(-\frac{\theta}{\sigma^2}-\sqrt{\frac{\theta^2}{\sigma^4}+\frac{2}{\sigma^2\nu}};-\frac{\theta}{\sigma^2}+\sqrt{\frac{\theta^2}{\sigma^4}+\frac{2}{\sigma^2\nu}}\right)=(-9{,}05;11{,}05)$. There are two points $u=-\frac{\theta}{\sigma^2}-\frac{1}{2}=0{,}5$, $u+1=1{,}5$ in the range of definition such that the values of the moment generating function at these points are the same $M(u)=M(u+1)=0{,}996$. In the second case the moment generating function (5) range of definition is $\left(-\frac{\theta}{\sigma^2}-\sqrt{\frac{\theta^2}{\sigma^4}+\frac{2}{\sigma^2\nu}};-\frac{\theta}{\sigma^2}+\sqrt{\frac{\theta^2}{\sigma^4}+\frac{2}{\sigma^2\nu}}\right)=(-0{,}1;0{,}1)$. However the condition $\sqrt{\frac{\theta^2}{\sigma^4}+\frac{2}{\sigma^2\nu}}>\frac{1}{2}$ fails and $u+1=0{,}5$ doesn't belong to the moment generating function (5) range of definition.

2 Normal inverse Gaussian process and the variance gamma process simulation

The normal inverse Gaussian process and the variance gamma process are special cases of generalized hyperbolic process. The generalized hyperbolic distribution is defined in [5] through its characteristic function

$$\varphi(u) = \exp(iu\mu)\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+u)^2}\right)^{\lambda/2}\frac{K_\lambda(\delta\sqrt{\alpha^2-(\beta+u)^2})}{K_\lambda(\delta\sqrt{\alpha^2-\beta^2})},$$

where $K_\lambda(z) = \frac{1}{2}\int_0^\infty u^{\lambda-1}\exp\left(-\frac{1}{2}z(u+u^{-1})\right)du$ is the modified Bessel function, $z > 0$, $\mu \in \mathbb{R}$

$$\delta \geq 0, |\beta| < \alpha, \text{если}\, \lambda > 0$$

$$\delta > 0, |\beta| < \alpha, \text{если}\, \lambda = 0,$$

$$\delta > 0, |\beta| \leq \alpha, \text{если}\, \lambda < 0.$$

The generalized hyperbolic distribution depends in five parameters: α determines the shape, β determines the skewness, δ is a scaling parameter, μ determines the location and λ characterizes certain sub-classes. For $\lambda = -\frac{1}{2}$ we obtain the normal inverse Gaussian distribution $NIG(\alpha,\beta,\delta,\mu)$. For $\lambda = \sigma^2/\nu, \delta \to 0$ we obtain the variance gamma distribution $VG(\sigma,\nu,t)$ taking $\alpha = \sqrt{2/\nu + \theta^2/\sigma^4}, \beta = \theta/\sigma^2$.

The normal inverse Gaussian process and the variance gamma process simulations are considered in [6, 7]. We propose another algorithm that is simpler and universal. The superposition method [8] using is possible for the generalized hyperbolic process and its special cases simulation because of the generalized hyperbolic distribution density form [7]:

$$f_{\lambda,\alpha,\beta,\delta,\mu}(x) = \int_0^\infty f^N_{\beta y+\mu,y}(x) f^{GIG}_{\lambda,\delta,\sqrt{\alpha^2-\beta^2}}(y)dy, \quad (10)$$

where $f^N_{\beta y+\mu,y}(x)$ is the normal distribution density with mean $\beta y+\mu$, and variance y, $f^{GIG}_{\lambda,\delta,\sqrt{\alpha^2-\beta^2}}(y)$ is the generalized inverse Gaussian distribution density with the parameters $\lambda, a = \delta, b = \sqrt{\alpha^2-\beta^2}$ [5, 9].

The algorithm contains the following sequence of steps.

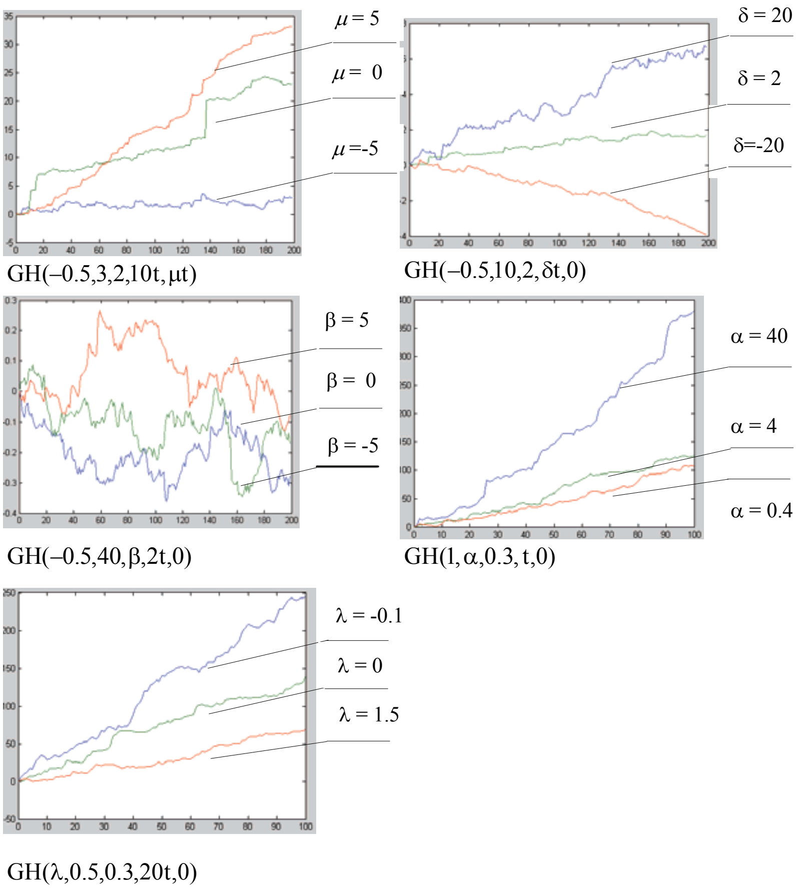

Figure 1. The sample paths of generalized hyperbolic processes

Algorithm

1) Generate independent generalized inverse Gaussian random numbers $\{\xi_k, k \geq 1\}$ with the parameters $\lambda, a = \delta\Delta t, b = \sqrt{\alpha^2 - \beta^2}$ [9]:

$$\xi_k \sim GIG(\lambda, \delta\Delta t, \sqrt{\alpha^2 - \beta^2}),\ k \geq 1.$$

where $\Delta t > 0$.

2) Generate independent generalized hyperbolic random numbers $\{\eta_k, k \geq 1\}$ as normal random numbers with the parameters $E\eta_k = \beta\xi_k + \mu\Delta t$, $D\eta_k = \xi_k$:

$$\eta_k \sim N(\beta\xi_k + \mu\Delta t, \xi_k),\ k \geq 1.$$

3) Generate generalized hyperbolic process $L^{GH} = \left(L_t^{GH}\right)_{t\geq 0}$ with the parameters $\lambda, \alpha, \beta, \delta, \mu$ as

$$L_0^{GH} = 0,\ L_{k\Delta t}^{GH} = L_{(k-1)\Delta t}^{GH} + \eta_k,\ k \geq 1.$$

The normal inverse Gaussian process is simulated using the above algorithm when the parameter $\lambda = -\frac{1}{2}$. The variance gamma process is simulated using the above algorithm when $\lambda = \sigma^2 / \nu, \delta \to 0$, $\alpha = \sqrt{2/\nu + \theta^2/\sigma^4}, \beta = \theta/\sigma^2$.
The sample paths of generalized hyperbolic processes are presented at the figure 1. The generalized hyperbolic processes simulations are carried out in MATLAB® 7.6.0 (R2008a).

The values of the Levy processes parameters for the existence of Esscher transform are deduced at the paper. A new algorithm of normal inverse Gaussian process and variance gamma process are considered.

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