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GARCH(1,1) models with stable residuals

Abstract. The focus of this paper is the use of stable distributions for GARCH models. Such models are applied for the analysis of financial and economic time series, which have several special properties: volatility clustering, heavy tails and asymmetry of residuals distributions. Below we compare the properties of stable and tempered stable distributions and describe methodologies for constructing models and subsequent estimation of parameters using the maximum likelihood method. We also analyze an example of building models on real data in order to illustrate that tempered stable distributions could be used in financial time series models. Moreover, such distributions can show better results in comparison with traditionally used distributions.

Keywords. GARCH model, stable distribution, tempered stable distribution, maximum likelihood method

1. Introduction

GARCH model (Generalized Autoregressive Conditionally Heteroskedastic) is one of the most popular tools for economic and financial time series analysis. Such time series have a number special features to be taken into consideration while modelling. Heavy tails of residuals distribution point out at outliers with very high values. Distribution can also be non-symmetric because various news and events influence financial assets prices in different ways. Volatility clustering means that small changes of financial assets prices are followed by small changes and bigger changes are followed by bigger ones.

Many studies established the fact that GARCH model with normal residuals distribution produces unsatisfactory results. The reason for this is that normal distribution is symmetric and its tails are not heavy enough. Thus, there exists actual problem of analysis of GARCH models with different residuals distributions (different from normal). For example, in recent years models with α-stable distributions has been studied [2-4]. Such models produce fine results but still have several disadvantages. Firstly, α-stable distributions have finite moments, only for orders less than α. Secondly, α-stable distributions have an explicit form of the distribution density function only in a few particular cases (α = 0.5, α = 1, α = 2). Thirdly, the tails of α-stable distributions turn out to be too heavy and insufficiently effective for real data analysis. To avoid the difficulties described above, various generalizations of α-stable distributions have been developed, which form a class of distributions called tempered stable.

In this paper we analyze GARCH(1,1) models with α-stable, classical tempered stable (introduced by Koponen [5]), modified tempered stable and Kim-Rachev (introduced by Kim et al. in [6] and [7] respectively) distributions and consider an example of constructing GARCH (1,1) models with different distributions using real data and the choice of the optimal model for forecasting based on statistical criteria.

2. GARCH(1,1) model with stable distributions

2.1. α-stable distribution

Let $\alpha \in (0,2]$, $\beta \in [-1,1]$, $\sigma \in (0,+\infty)$, $\mu \in$ R. Then random variable X has α-stable distribution if its characteristic function has the following form

$$\Phi_X(u) = \Phi_{stable}(u;\alpha,\sigma,\beta,\mu) = E\left[e^{iuX}\right] = \begin{cases} \exp\left(i\mu u - |\sigma u|^{\alpha}\left(1 - i\beta(sign\, u)\tan\dfrac{\pi\alpha}{2}\right)\right), & \alpha \neq 1 \\ \exp\left(i\mu u - |\sigma u|\left(1 + i\beta\dfrac{2}{\pi}(sign\, u)\ln|u|\right)\right), & \alpha = 1 \end{cases}$$

where $u \in$ R,

$$sign\, t = \begin{cases} 1, & t > 0, \\ 0, & t = 0, \\ -1, & t < 0. \end{cases}$$

We denote $X \sim S_\alpha(\sigma,\beta,\mu)$. Parameters have the following meaning: α – stability; β – asymmetry; μ – shift; σ – scale. If we set μ=0, σ=1, β=0, then we have standard α-stable distribution and denote $stdS_\alpha$. We note several properties of the α-stable distribution:

$$E|X| = \mu,\ \alpha > 1;\ E|X| = \infty,\ 0 < \alpha \le 1;\ E|X|^p < \infty,\ 0 < p < \alpha;\ E|X|^p = \infty,\ p \ge \alpha .$$

The above properties cause certain difficulties in the use of α-stable distribution as GARCH model residuals. This is described in more detail in [4] and [8].

2.2. Classical tempered stable distribution

Let $\alpha \in (0,1) \cup (1,2), \sigma, \lambda_+, \lambda_- > 0, \mu \in$ R. Then random variable X has classical tempered stable distribution (CTS) if its characteristic function has the following form

$$\Phi_X(u) = \Phi_{CTS}(u;\alpha,\sigma,\lambda_+,\lambda_-,\mu) =$$
$$= \exp\left(iu\mu - iu\sigma\Gamma(1-\alpha)\left(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1}\right) + \sigma\Gamma(-\alpha)\left((\lambda_+ - iu)^\alpha - \lambda_+^\alpha + (\lambda_+ - iu)^\alpha - \lambda_-^\alpha\right)\right),$$

where $u \in$ R, Γ – gamma function. We denote $X \sim CTS(\alpha,\sigma,\lambda_+,\lambda_-,\mu)$. The cumulants for the distribution are calculated as follows:

$$c_1(X) = \mu ,$$

$$c_n(X) = \alpha\Gamma(n-\alpha)\left(\lambda_+^{\alpha-n} + (-1)^n \lambda_-^{\alpha-n}\right),\ n = 2,3,\ldots .$$

Parameters α, μ, σ have the same meaning as in α-stable distribution. Parameters λ_+ and λ_- control the decay rate for the positive and negative tail, respectively. If $\lambda_+ > \lambda_-$ $(\lambda_+ < \lambda_-)$ then distribution has left (right) asymmetry, and if $\lambda_+ = \lambda_-$ then distribution is symmetric. Parameters λ_+, λ_- and α also determine the heaviness of the distribution tails. If we set

$$\sigma = \left(\Gamma(2-\alpha)\left(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2}\right)\right)^{-1}$$

then random variable $X \sim CTS(\alpha,\sigma,\lambda_+,\lambda_-,0)$ has zero mean, and variance is equal to 1. In these cases we say that X has standard CTS distribution with parameters α, λ_+, λ_- and denote $X \sim stdCTS(\alpha,\lambda_+,\lambda_-)$.

2.3. Modified tempered stable distribution

Let $\alpha \in (0,1) \cup (1,2), \sigma, \lambda_+, \lambda_- > 0, \mu \in$ R. Then random variable X has modified tempered stable distribution (MTS), if its characteristic function has the following form

$$\Phi_X(u) = \Phi_{MTS}(u;\alpha,\sigma,\lambda_+,\lambda_-,\mu) =$$
$$= \exp\left(iu\mu - \sigma\left(G_R(u;\alpha,\lambda_+) + G_R(u;\alpha,\lambda_-)\right) + iu\sigma\left(G_I(u;\alpha,\lambda_+) + G_I(u;\alpha,\lambda_-)\right)\right),$$

where for $u \in$ R,

$$G_R(\mathrm{x};\alpha,\lambda) = 2^{-\frac{\alpha+3}{2}}\sqrt{\pi}\Gamma\left(-\frac{\alpha}{2}\right)\left(\left(\lambda^2 + x^2\right)^{\frac{\alpha}{2}} - \lambda^\alpha\right),$$

$$G_I(\mathrm{x};\alpha,\lambda) = 2^{-\frac{\alpha+3}{2}}\Gamma\left(\frac{1-\alpha}{2}\right)\lambda^{\alpha-1}\left[{}_2F_1\left(1,\frac{1-\alpha}{2};\frac{3}{2};-\frac{x^2}{\lambda^2}\right) - 1\right],$$

where ${}_2F_1$ – hypergeometric function. The cumulants for the distribution are calculated as follows:

$$c_1(X) = \mu,$$

$$c_n(X) = 2^{n-\frac{\alpha+3}{2}}\sigma\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n-\alpha}{2}\right)\left(\lambda_+^{\alpha-n} + (-1)^n\lambda_-^{\alpha-2}\right),\ n = 2,3,\ldots .$$

We denote $X : MTS(\alpha,\sigma,\lambda_+,\lambda_-,\mu)$. Parameters has the same meaning as in CTS distribution. If we set

$$\sigma = 2^{\frac{\alpha+1}{2}}\left(\sqrt{\pi}\Gamma\left(1-\frac{\alpha}{2}\right)\left(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2}\right)\right)^{-1},$$

then random variable $X : MTS(\alpha,\sigma,\lambda_+,\lambda_-,0)$ has zero mean, and variance is equal to 1. In these cases we say that X has standard MTS distribution with parameters α, λ_+, λ_- and denote $X : stdMTS(\alpha,\lambda_+,\lambda_-)$.

2.4. Tempered stable Kim-Rachev distribution

Let $\alpha \in (0,1)\cup(1,2)$, $k_+,k_-,r_+,r_- > 0$, $p_+,p_- \in \{p > -\alpha \mid p \neq -1, p \neq 0\}$ and $\mu \in$ R. Then random variable X has Kim-Rachev tempered stable distribution (KRTS), if its characteristic function has the following form

$$\phi_X(u) = \phi_{KRTS}(u;\alpha,k_+,k_-,r_+,r_-,p_+,p_-,\mu) =$$
$$= \exp\left(iu\mu - iu\Gamma(1-\alpha)\left(\frac{k_+r_+}{p_+ + 1} - \frac{k_-r_-}{p_- + 1}\right) + k_+H(iu;\alpha,r_+,p_+) + k_-H\left(iu;\alpha,r_-,p_-\right)\right),$$

where $u \in$ R, Γ – gamma function,

$$H(x;\alpha,r,p)=\frac{\Gamma(-\alpha)}{p}\left({}_2F_1(p,-\alpha;1+p;rx)-1\right),$$

where ${}_2F_1$ – hypergeometric function. The cumulants for the distribution are calculated as follows:

$$c_1(X)=\mu,$$

$$c_n(X)=\Gamma(n-\alpha)\left(\frac{k_+r_+^n}{p_++n}+(-1)^n\frac{k_-r_-^n}{p_-+n}\right), n=2,3,\dots .$$

We denote $X \sim KRTS(\alpha,k_+,k_-,r_+,r_-,p_+,p_-,\mu)$. If we set

$$k_+=C\frac{\alpha+p_+}{r_+^\alpha},\ k_-=C\frac{\alpha+p_-}{r_-^\alpha},$$

where

$$C=\frac{1}{\Gamma(2-\alpha)}\left(\frac{\alpha+p_+}{2+p_+}r_+^{2-\alpha}+\frac{\alpha+p_-}{2+p_-}r_-^{2-\alpha}\right)^{-1},$$

then random variable $X \sim KRTS(\alpha,k_+,k_-,r_+,r_-,p_+,p_-,0)$ has zero mean, and variance is equal to 1. In these cases we say that X has standard $KRTS$ distribution with parameters α,r_+,r_-,p_+,p_- and denote $X \sim stdKRTS(\alpha,r_+,r_-,p_+,p_-)$.

2.5. GARCH(1,1) model

Process $X_t,\ t\in Z$ satisfies $GARCH(1,1)$ model (Generalized Autoregressive Conditionally Heteroskedastic) if

$$\begin{cases} X_t=\sigma_t Z_t, \\ \sigma_t^2=\omega_0+\omega_1 X_{t-1}^2+\omega_2\sigma_{t-1}^2, \end{cases}$$

where $\{Z_t, t\in Z\}$ – independent identically distributed random variables and $\omega_0>0,\omega_1>0,\omega_2>0$ – model parameters. The condition of stationarity has the following form:

$$\omega_1+\omega_2<1.$$

2.6. Parameters estimates

We specify the parameters vectors and the range of parameters for the GARCH (1,1) models with different distributions of Z_t. Let $Z_t \sim \mathrm{N}(0,1)$. Then vector of the parameters has the following form

$$\theta_{norm} = (\omega_0, \omega_1, \omega_2)^T,$$

and the range of the parameters is

$$K_{norm} = \{\theta_{norm} : \omega_1 + \omega_2 < 1; 0 < \min\{\omega_0, \omega_1, \omega_2\} \le \max\{\omega_0, \omega_1, \omega_2\} < 1\}.$$

For the case when $Z_t \sim S_\alpha(\beta, \sigma, \mu)$, we define a parameters vector and the range of parameters as follows:

$$\theta_{stable} = (\omega_0, \omega_1, \omega_2, \alpha, \beta, \sigma, \mu)^T,\ K_{stable} = \begin{Bmatrix} \theta_{stable} : \omega_1 + \omega_2 < 1; 0 < \min\{\omega_0, \omega_1, \omega_2\} \le \max\{\omega_0, \omega_1, \omega_2\} < 1; \\ \alpha \in (0,2];\ \beta \in [-1,1];\ \sigma \in (0, +\infty) \end{Bmatrix}.$$

For the case when $Z_t \sim CTS(\alpha, \sigma, \lambda_+, \lambda_-, \mu)$:

$$\theta_{CTS} = (\omega_0, \omega_1, \omega_2, \alpha, \lambda_+, \lambda_-, \sigma, \mu)^T,\ K_{CTS} = \begin{Bmatrix} \theta_{CTS} : \omega_1 + \omega_2 < 1; 0 < \min\{\omega_0, \omega_1, \omega_2\} \le \max\{\omega_0, \omega_1, \omega_2\} < 1; \\ \alpha \in (0,1) \cup (1,2);\ \lambda_+, \lambda_-,\ \sigma \in (0, +\infty) \end{Bmatrix}.$$

For the case when $Z_t \sim MTS(\alpha, \sigma, \lambda_+, \lambda_-, \mu)$:

$$\theta_{MTS} = (\omega_0, \omega_1, \omega_2, \alpha, \lambda_+, \lambda_-, \sigma, \mu)^T,$$
$$K_{MTS} = \begin{Bmatrix} \theta_{MTS} : \omega_1 + \omega_2 < 1; 0 < \min\{\omega_0, \omega_1, \omega_2\} \le \max\{\omega_0, \omega_1, \omega_2\} < 1; \\ \alpha \in (0,1) \cup (1,2);\ \lambda_+, \lambda_-, \sigma \in (0, +\infty) \end{Bmatrix}.$$

For the case when $Z_t \sim KRTS(\alpha, k_+, k_-, r_+, r_-, p_+, p_-, \mu)$:

$$\theta_{KRTS} = (\omega_0, \omega_1, \omega_2, \alpha, k_+, k_-, r_+, r_-, p_+, p_-, \mu)^T,$$
$$K_{KRTS} = \begin{Bmatrix} \theta_{KRTS} : \omega_1 + \omega_2 < 1; 0 < \min\{\omega_0, \omega_1, \omega_2\} \le \max\{\omega_0, \omega_1, \omega_2\} < 1; \\ \alpha \in (0,1) \cup (1,2);\ k_+, k_-, r_+, r_- \in (0, +\infty); \\ p_+, p_- \in \{p > -\alpha \mid p \ne -1, p \ne 0\} \end{Bmatrix}.$$

We denote:

$$\theta = \begin{cases} \theta_{norm}, Z_t : N(0,1) \\ \theta_{stable}, Z_t : S_\alpha(\beta,\sigma,\mu) \\ \theta_{CTS}, Z_t : CTS(\alpha,\sigma,\lambda_+,\lambda_-,\mu) \\ \theta_{MTS}, Z_t : MTS(\alpha,\sigma,\lambda_+,\lambda_-,\mu) \\ \theta_{KRTS}, Z_t : KRTS(\alpha,k_+,k_-,r_+,r_-,p_+,p_-,\mu) \end{cases}$$

$$K = \begin{cases} K_{norm}, Z_t : N(0,1) \\ K_{stable}, Z_t : S_\alpha(\beta,\sigma,\mu) \\ K_{CTS}, Z_t : CTS(\alpha,\sigma,\lambda_+,\lambda_-,\mu) \\ K_{MTS}, Z_t : MTS(\alpha,\sigma,\lambda_+,\lambda_-,\mu) \\ K_{KRTS}, Z_t : KRTS(\alpha,k_+,k_-,r_+,r_-,p_+,p_-,\mu) \end{cases}$$

To estimate the parameters of GARCH(1,1) model we use the maximum likelihood method. Suppose we have a sample of length n $X_1,\ldots,X_n$, $n \in \mathrm{N}$ behind the process $X_t, t \in \mathrm{Z}$. Then estimate $\hat{\theta}_n$ of parameters vector θ of GARCH(1,1) model on a compact set K is defined as follows:

$$\hat{\theta}_n = \arg\max_{\theta \in K} \hat{L}_n(\theta),$$

where

$$\hat{L}_n(\theta) = \frac{1}{n}\sum_{t=1}^{n} ln\left[\frac{1}{\hat{h}_t(\theta)} f\left(\frac{X_t}{\hat{h}_t(\theta)}\right)\right],$$

$\theta \in K$, $t = \overline{1,n}$, $f(x)$ – density distribution function of Z_t and $\hat{h}_t(\theta)$ can be considered as estimate of σ_t. The choice of form of log likelihood function $\hat{L}_n(\theta)$ described in [8]. Note that $\hat{h}_t(\theta_0) = \sigma_t$ for all $t \in \mathrm{N}$, a θ_0 – vector of true values of the parameters. For $\hat{h}_t(\theta)$ we use function $\hat{h}_t(\theta) = \hat{y}_t^{\frac{1}{2}}(\theta)$, where $\hat{y}_t(\theta)$ has the following form:

$$\hat{y}_t(\theta) = \begin{cases} \varepsilon, t = 0 \\ w_0 + w_1 X_{t-1}^2 + w_2 \hat{y}_{t-1}(\theta), t \geq 1, \end{cases}$$

$\varepsilon \in [0, \infty)$ – random initial value. We note that for the distributions under consideration most often the explicit form of the distribution density function is unknown. To find the values of the function $f(x), x \in \mathrm{R}$ we use the inverse Fourier transform of the characteristic function $\phi(t)$. For practical experiments we use R language with additional packages: MixedTS, SymTS, fourierin, GEVStableGarch.

3. Real data empirical for the GARCH(1,1) model

We consider an example of constructing GARCH(1,1) using data of Intel (INTC) stock prices. We use daily data for the period from the 1st of November 2007 till the 1st of November 2017. Suppose that the dynamics of the logarithm of return rate of securities has the following form:

$$\log\left(\frac{S_t}{S_{t-1}}\right) = \sigma_t Z_t,\ 1 \le t \le T.$$

$S_t > 0$ – security price at the time moment $t, t \in \mathrm{N}$; $\sigma_t Z_t$ – return volatility, defined using the GARCH(1,1) model. For the time series we construct GARCH(1,1) models with normal, α-stable, classical tempered stable, modified tempered stable, KR residuals distributions. Results of estimation are shown in Table 1.

Table 1. Parameter estimates of GARCH(1,1) models with different residuals distributions for INTC data.

	ω_0	ω_1	ω_2	α	λ_+	λ_-
GARCH(1,1)-normal	6.8556E-5	0.2133	0.8589	-	-	-
GARCH(1,1)-stable	5.2925E-5	0.1371	0.9316	1.7131	-	-
GARCH(1,1)-CTS	5.3893E-5	0.1252	0.8122	1.8268	0.0745	0.0779
GARCH(1,1)-MTS	6.2704E-5	0.1423	0.7453	1.7779	0.0775	0.0746

To analyze how the model fits time series we use Kolmogorov-Smirnov test. Null hypotheses are defined as follows: $H_0(normal)$, $H_0(stable)$, $H_0(CTS)$, $H_0(MTS)$, $H_0(KRTS)$ – the residuals correspond to normal, α-stable, classical tempered stable, modified tempered stable, KR residuals respectively. Kolmogorov-Smirnov statistics and its p-values are shown in Table 2. According to the results obtained, the null hypothesis $H_0(normal)$ is rejected, because The p-value is less than the significance level of 5%. The remaining null hypotheses are not rejected.

Table 2. Kolmogorov-Smirnov statistics and p-values

Residuals distribution	KS	p-value
GARCH(1,1)-normal	0.1291	0.0000
GARCH(1,1)-stable	0.0308	0.0603
GARCH(1,1)-CTS	0.0209	0.1554
GARCH(1,1)-MTS	0.0226	0.1614
GARCH(1,1)-KRTS	0.0137	0.4926

To compare models and select the best we used the following information criteria:

Akaike information criterion – AIC

$$AIC = -2\frac{LLF}{T} + 2\left(\frac{k}{T}\right),$$

Bayesian information criterion – BIC,

$$BIC = -2\frac{LLF}{T} + \frac{k \ln T}{T},$$

Hannan-Quinn information criterion – HQIC

$$HQIC = -2\frac{LLF}{T} + \frac{2k \ln(\ln T)}{T}.$$

where LLF – log likelihood function value, T – time series length, k – number of model parameters. The best is the model that has the lowest value of the information criterion. According to the results presented in Table 3, it should be recognized the best model GARCH(1,1)-KRTS.

Table 3. Information criteria results

	GARCH(1,1)-	GARCH(1,1)-	GARCH(1,1)-	GARCH(1,1)-	GARCH(1,1)-

	normal	stable	CTS	MTS	KRTS
AIC	-3.8667	-4.2782	-4.3948	-4.4420	-4.6103
BIC	-3.8594	-4.2691	-4.3820	-4.4618	-4.5819
HQIC	-3.8641	-4.2750	-4.3903	-4.4628	-4.5957

4. Conclusion

In this paper we compared GARCH(1,1) models with different residuals distributions. GARCH(1,1)-normal model was rejected by Kolmogorov-Smirnov test, while models with α -stable and tempered stable residuals were not rejected. We also checked relative quality of the models using information criteria. The reason for the good statistical results for tempered stable GARCH(1,1) models is that skewness and fat-tail property of their innovation are taken into account.

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